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PAPER

A General View about Games Abstention, Absence, Alternatives

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Abstract In this paper we introduce a new and more general kind of cooperative games in which players have many alternatives to choose, or many goals to try to realize, but also have the possibility of abstaining or of being absent for a while. In these games we define coalitions and study their properties observing how it is possible to obtain all the classical definitions of cooperative game theory inside this new setting.

Keywords Cooperative Games, Characteristic Function, Alternatives, Abstention, Absence.

JEL Classification C71, D71

Introduction

In classical cooperative game theory it is assumed that every player tries to win, that is: it is assumed that every player makes all possible efforts in order to obtain the result he/she wants, and it is also assumed that every other behaviour (i.e. abstention, absence or partial efforts) can not give the same advantages to the player. By this some author says simply that abstention is not a good choice for a player; the same happens for absence. The same way of reasoning says, in practice, that there are only two alternatives in the final result of the game: the best possible result (win) and all the other possible results resumed in one single event (loss); moreover, every consideration is so concentrated in the first result (win) that the second possibility is never explicitly expressed. By this assumption, every definition and property of cooperative games follows. In particular, it follows that a coalition is simply a subset of the set of players.

If we look at what happens in reality, for instance in a parliament in the moment of a ballot, we see that players in the same coalition (in this case parties) may behave in different ways: a party in a coalition can vote in favour of the law at ballot, but also can abstain (i.e. it is present but does not vote), or it can be absent from the game (for instance leaving the parliament for a while). And we must say that this last possibility has been used quite often in the Italian parliament. So they have many ways of expressing some partial and/or temporary disagreement with the rest of the coalition. Absence and abstention are really used by players in the game to obtain some kinds of advantages, that may be: long term advantages, side payments, and so on. Absence and abstention can be considered in the classical case only taking into account all the (classical) games that can be obtained starting from a fixed game,

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assuming that a voter abstains or is absent. Analogously for direct or indirect shareholders of a corporation, where, in a shareholders' meeting a shareholder may be absent, or present, and, in this case, it can abstain or vote in some way.

On the other side, it is possible that the result of the game may be not only the victory of a group of players and the loss of another group: for instance we may have a tie. And in other cases there may be many alternatives a player can choose, as we will discuss later.

The possibility of abstention has been considered first some times ago, by Fishburn (Fishburn 1973), but in a different context, see also (Felsenthal and Machover 1997) where Fishburn's considerations have been remade. After, it was always excluded, and it has been considered again only recently as a distinct alternative, see hereafter. Roughly speaking: all the authors considered abstention as an 'unreasonable' alternative for a player and so discarded, because they assumed that power, or any other kinds of advantages, can be achieved only with active playing. 'But authors do not even bother to supply this rationalisation. More surprising still: when dealing with real-life decision rules […] most authors mis-report the rules as though abstention were not a distinct option' (Felsenthal and Machover 1998, p.22). This mis-representation is due to the assumption just mentioned that leads to 'what philosophers of sciences have called theory-laden or theory-biased observation - a common occurrence, akin to optical illusion, whereby an observer's perception is unconsciously distorted so as to fit a preconception' (Felsenthal and Machover 1998, p. 280).

Indeed, sometimes it is possible to identify abstention with, for instance, a 'no vote', e.g. in a parliament where a motion is approved only if the majority votes yes, but, in a more correct way, abstention must be represented as a distinct alternative.

For other discussions about abstention see comment 2.2.4 together with the discussion at the beginning of chapter 8 in (Felsenthal and Machover 1998), and (Felsenthal and Machover 2001).

But the possibility of absence is absent from the most part of papers too! If we look at the papers on the subject we see also that absence is nearly never considered. Unique exceptions: 1) Felsenthal and Machover 2001: in a note at p.88, they say that they consider abstention in a 'wide sense', i.e. also as the possibility of absence. 2) Freixas and Zwicker 2003, p.401, note 1: 'But a voter may abstain for a host of other reasons -such as absence […]-.' So identifying absence with abstention in most cases. Noting just after that: 'An exception […] arises when a voting system includes a provision for a quorum. [...] in such a case it may prove impossible to identify an absence with an abstention [...]'. Indeed absence changes the game, and so it seems not possible to consider it as an alternative of the same kind of the other ones.

On the other side, it has been observed that a player may have many alternatives to choose, for instance when players have to choose among many candidates for some place/ job. Indeed there are several papers considering games in which players have this possibility, see for instance the papers of: Bolger, Amer et al., Freixas et al., Felsenthal and Machover in the bibliography. In these papers abstention at the vote, that is in the input of the game, can be considered as one of these alternatives, as well as a tie in the output. The aim of these papers is to define and characterize some sort of power index, using the order structure on the alternatives when this is present, in particular in Freixas et al., and in Felsenthal and Machover papers.

Roughly speaking, in these papers we have a set of indices contained in the set of integers together with the natural order on the indices: these indices indicate the various choices (or alternatives) a player has, and/ or the "quantity of effort" the player puts in his choice. One of the choice stands for abstention, usually 0. Due to the structure of the choices it is not possible to consider the possibility of absence, since absence is not (simply) another alternative or another behaviour; absence is a completely different alternative and a completely different behaviour.

Absence of a player changes the game into another different game since the absent player has gone away. So, this possibility must be taken into consideration in another way.

In this paper we take abstention seriously and we consider it as a reasonable behaviour of a voter. Moreover, we consider reasonable the possibility of absence too. So, we introduce games in which players have many alternatives to choose but also the possibility of abstention and/or absence. Then, generalizing what has been done so far, we introduce coalitions in this new setting in a way that is very similar to the way in which coalitions are considered in a parliament, i.e. in a very general sense.

In future papers we will introduce a power measure in a Banzhaf - Colemann style showing some of its properties, and we will show examples of particular applications of the games introduced here. Moreover, we will extend what is introduced here to the case of indirect control of corporations, see Gambarelli and Owen (1994) and Denti and Prati (2001) and (2004).

In section 1 we show an introductory example together with some discussions. In 2 we give the definitions of cooperative games in which players have many alternatives to choose, but also have the possibility of abstaining or of being absent for a while. In 3 we re-build classical games in the new setting.

1. Initial Examples and Observations

Throughout the paper we try to stick to the terminology of Felsenthal and Machover (1998). We begin considering some examples of classical weighted voting games, hereafter wvg, where bills are passed with simple majority.

1.1) Example Part I (and Definition)

A board has three voters A, B, C, with weights respectively: 3, 2, 1. A bill is passed with simple majority. Here the voters have only two alternatives: a player can join a certain coalition/group to try to win, or join the other coalition/group, i.e. the complementary one, again to try to win, and prevent, in the same time, the victory of the first group.

a) If all the voters are present and 'active', the majority quota is $q=4$, and the game is: G=[4; 3,2,1]. We denote this game as the *principal game*.
b) Suppose voter C decides to abstain: then we obtain the following wvg derived by the principal game G: G_1=[4; 3,2,0].
c) Now suppose C leaves the board (i.e. the game): then we obtain another wvg derived by the principal game: G_2=[3; 3,2,Ø] where we indicate with Ø the fact that player C is absent. Naturally this game coincides with the game [3;3,2], and in this last game we have that A is a dictator on the contrary of the previous case.

So, starting from the principal game we can obtain several derived games considering all the possibilities the three players have: be active, abstain, be absent. In particular, for the game with only three voters in this example we have then 26 different derived games (with the empty game): together with G_1 and G_2 seen above we have G_3=[4; 3,0,1], G_4=[2; 3,Ø,1], G_5 =[4; 0,2,1], G_6=[2; Ø,2,1], G_7=[4; 3,0,0], G_8=[2; 3,0,Ø], and so on.

So we have seen in a practical case what has been observed by Freixas and Zwicker (2003) in the quotation above: abstention and absence change the game. Moreover, it seems not possible to consider absence as an 'alternative' in the same way as abstention; using now the word coalition in an intuitive sense, a coalition of voters has different possibilities of winning in the three cases, i.e.: a) when all the voters are present in the game and active; b) when some of them abstains; c) when someone is missing.

If we look at the example above we see that the coalition {A,B} wins in several different cases:

- when the two members of the coalition are active: see (for instance) G;
- when some voter is abstaining: see G_1;
- when some voter is absent: see G_4.

For other considerations and examples of games derived by a principal game, see Prati (2002). About wvg's we observe the following, leaving proof and counterexamples to the reader:

2) Observation

Let *N= {1, 2, ... , n}* be the set of players/voters (so *N* is also the *grand coalition*), then take the simple majority wvg *G= [q; w1, ... , wn]*. If the coalition *C* is winning in *G*, and $1 \notin C$, then C is winning in the two simple majority wvg's obtained by *G* when voter 1 abstains or is missing, while, if $1 \notin C$ and *C* is winning in the two derived games, then $C \cup \{1\}$ is still winning in these two games. If *C* is winning in the game obtained by *G* when 1 abstains, and $1 \notin C$, then *C* is winning also in *G*. The same happens for all the other players.

Voters in a board may have many alternatives to choose, and voters in a coalition may be more or less active, i.e. may decide to abstain or leave the game for a while.

In our opinion, it seems better to distinguish alternatives in the input of the game from alternatives in the output, as done in Freixas et al. papers, whose works generalize some of the preceding ones. See also the part 'Background and summary' in Freixas and Zwicker (2003) for some other hints in the comparison among papers on games with alternatives.

2. Generalizing Classical Cooperative Games

3) Definition and observations

Fix:

1) *N = {1, 2, ..., n}*, the set of voters / members/ players;
2) *AL = {A1, A2, ... , Am}*, with *m>2*, the set of alternatives or goals;
3) two symbols A and Ø, that do not belong to AL.

Then define:

a) AL' = ALÈ{A, Ø};
b) a situation, w.r.t. the set AL, is an n-tuple (s1, ... , sn) such that: for every i, siÎAL'.
c) If $(s_1, ... , s_n)$ is a situation, then:
   c.1) s_i=A_k should be interpreted as 'voter i is active and chooses alternative Ak';
   c.2) s_i= A as 'i is present but abstains';
   c.3) s_i= Ø as 'i is absent'.
d) If s_i≠A,Ø we say that voter *i* is active in the situation.
e) If s_i≠Ø we say that voter *i* is present.
f) If *S* is a situation then Ac(*S*)={ $i \in N$: *i* is active in *S*}, Pr(*S*)={ $i \in N$: *i* is present in *S*}.
g) Sit is the set of all the situations w.r.t. AL'.

4) Example (Part II)

In the wvg of example 1, with three voters A, B, C, and weights, 3, 2, 1, suppose there are two alternatives that are respectively A1 and A2, that can be interpreted, depending on the context, as: vote in favour of a law or contrary to it, or vote for candidate 1 or for candidate 2, or … .
The situations where all the voters are active are, for instance: $S_1=(A_1, A_1, A_1)$, $S_2=(A_1, A_1, A_2)$, $S_3=(A_1, A_2, A_1)$, and so on. The situations where some player is not active are, for instance:

$T_1=(A_1, A_1, Ø)$, $T_2=(A_1, A_1, 0)$, $T_3=(A_1, 0, Ø)$, and so on.
Now: note that situations determine not only the classical game that is going to be played, but also the set of (classical) coalitions playing in the game too, since, for instance:

- by S_1 is derived the game [4;3,2,1] in which the coalition {A,B,C} is formed and tries to obtain alternative A1;
- by S_2 is derived the game [4;3,2,1] in which the two coalitions {A,B} and {C} are formed; here the coalition {A,B} tries to obtain alternative A_1, and the complementary coalition {C} tries to obtain alternative A_2.
- And so on, while:
- by T_1 is derived the game [3;3,2] in which the coalition {A,B} is formed and tries to obtain alternative A_1 (while C is absent).
- by T_2 is derived the game [4;3,2,0] in which the two coalitions {A,B} and {C} are formed; here the coalition {A,B} tries to obtain alternative A_1, while {C} abstains.
- by T_3 is derived the game [3;3,0] in which the two coalitions {A} and {B} are formed; here the coalition {A} tries to obtain alternative A_1, while {B} abstains.

And so on, if we look only to classical games; on the contrary, if we think in a more general way, and paying attention to what happens in real situations in a parliament, by T_1 we may think that the game is always the game [4;3,2,1] and (for instance) the coalition {A,B,C} is formed but the voter C has gone away from the game for a while.
In the same way, if we consider situation T_4=(A1,A2, Ø) we may think that the game is always [4;3,2,1], and the coalition {A,C} is formed together with the coalition {B}, but here the voter C has leaved the game for a while, may be to obtain some particular favour from A. If we think only in a classical way, in this situation we should think that we are considering the game [4;3,2], and that the two coalitions {A} and {B} are formed.

As we have seen in the example, a situation is then a "photograph" of what every voter is going to do in that particular moment of a "large and long" game. Following these ideas we give:

5) Definition

a) Fixed a situation $S=(s_1, \ldots , s_n)$, a <u>coalition</u> *C* in *S* is a subset of *N* such that: there is at most one alternative Ak such that:

1) for every $1 \in C$ then $s_i \in \{A_k, A, Ø\}$; and 2) for every $j \notin C$ then $s_j \in$ Al'-$\{A_k\}$.
That is: *C* is a coalition if every voter in *C* either chooses the same alternative A_k or abstains or is absent, while voters outside C choose some different alternative, but may abstain or be absent too.

b) Co(S)={C: C is a coalition in the situation S}; if $T \subseteq Sit$, Co(T)={*C: C* is a coalition in S and S∈T}.
c) By the notation C_h we denote a coalition such that: for every $i \in C_h$, then $s_i \in \{Ah, A, Ø\}$. And we say briefly that the coalition is <u>h-active</u>, i.e. tries to obtain alternative A_h.
d) Fixed *S*, a <u>partition of N in coalitions</u>, <u>w.r.t S</u>, is an *(m+1)*-tuple $P=(C_1,..,C_m,C_{m+1})$ such that for every h=1,..,m, C_h is h-active in S, and $C_{m+1}= N - Y_{j=1}^{m} C_j$. So, C_{m+1} is the coalition of voters that do not want to take sides, and, if $i \in C_{m+1}$, then s_i=A,Ø.
e) If $T \subseteq S_{it}$, Pc(T)={P: P is a partition of N in coalitions w.r.t. S and $S \in T$}.
f) If P∈Pc(T), and $P=(C_1, \ldots ,C_{m+1})$ and C∈Co(*S*) with S∈T, then C∈P stands for: there is h=1, .. ,m+1, such that $C_h=C$.

Therefore, a "*partition in coalitions*" is the formalization of the division of a board in different coalitions/ groups in a particular situation (instant), may be in the moment of a ballot or during some discussions. In Freixas et al. papers and in Felsenthal and Machover papers there is a total order on alternatives which makes possible to interpret the medium alternative (i.e. 0) as abstention. In all the other papers on games with many alternatives, there is no structure on the alternatives. In all these papers, abstention can be taken into consideration but only as a particular alternative and voters can form a coalition only if they choose the same alternative. That is: if voter $i \in C$ chooses alternative k, then all the other $j \in C$ cannot choose another alternative; they cannot abstain neither, since abstention is just another alternative. But there can exists the coalition of the abstaining voters.

In our setting coalitions can be classified as more or less active in their behaviour:

6) Definition

a) A coalition C in S strongly chooses A_k, if for every $i \in C$, then $s_i = A_k$.
b) a coalition C in S partially chooses A_k, if for every $i \in C$, then $s_i \in \{A_k, A\}$.
c) a coalition C in S improperly chooses A_k, if for every $i \in C$, then $s_i \in \{A_k, A, Ø\}$.

The more interesting (and probably "stronger") coalitions are the strongly k-active coalitions, i.e. when all the voters in the coalitions choose the same alternative A_k and they are not absent neither abstain.

In the following by S, or S', we will always denote a situation; by C or D, or C_k, D_k, a coalition, and by P, or P', a partition in coalitions w.r.t. a given situation S.

7) Example

In a town council there are five parties Right, Middle Right, Middle, Left, Extreme Left, briefly R, MR, M, L, EL; they elected respectively 2, 3, 5, 1, 1 members in the council. Three candidates to mayor are presented: r, m and l respectively by party R, M and L. The candidate who gets more votes from the councillors will be elected, and a member or a party can abstain or be absent. The two parties MR and M formed a coalition to govern the town and they always voted together, but now members of MR don't want to vote m because his program is partially in conflict with MR's program. Because of this, M proposed some favours to MR members if m will be elected. So MR members leave the council at the ballot. In the same time EL that presented no candidate abstains. At the end of the story, m is elected. Here N={R, MR, M, L, EL}, and AL={r, m, l}. In the moment of the ballot the situation is S=(r,Ø,m,l,A). In this situation possible coalitions at the ballot are, for instance:

- {R} which is r-active;
- {M,EL} which is partially m-active;
- {MR,M} which is improperly m-active;
- {MR,M,EL}, {EL}, {R,EL}, and so on.

The set {M,L} is not a coalition here. The situation (m,Ø,Ø,Ø,m) seems impossible to be realized in the council: the two parties at the extreme wings vote together the candidate of the middle party, so forming a coalition (in practice and also in our setting), while all the others are absent. The set {{R},{MR,M},{L},{EL}} is a partition in coalitions of the voters in the situation S=(r,Ø,m,l,A) described early. Another partition in the same situation is {{R}, {MR,M}, {L,EL}}, and so on.

8) Definition (and comments)

A <u>Game with Alternatives, Abstention, Absence</u> G, briefly <u>Gaaa</u>, is characterized by:
a) a set of voters N={1, ... , n};

b) a set of alternatives AL with cardinality $m \geq 2$, and the two symbols A, Ø;
c) a set Sit(G) of possible situations, contained in Sit;
d) a set Out(G) of possible outcomes, such that: 1) $\emptyset \in$ Out(G); 2) AL $\subseteq$ Out(G). An alternative in the input of the game should be interpreted also as a goal or a result that can be interesting for some voter. But there can be some result that has no interest for all the voters or that cannot be foreseen, and this result is in Out(G) – AL.
e) a function Final Result, FR, with domain Sit(G)×Pc(Sit(G)) (the cartesian product of Sit(G) and Pc(Sit(G))), and range Out(G). If:
e.1) FR(S,P)= Ø, this means that no result has been achieved: that is we have a tie;
e.2) FR(S,P)=A_k, this should be interpreted as: 'the alternative A_k is chosen as a consequence of situation S and the distribution of coalitions in P', or 'goal Ak is reached';
e.3) FR(S,P)$\in$Out(G)-AL, this means that the outcome FR(S,P) has been achieved, but this outcome was not expected by the voters.
f) a function V(S,C,P), characteristic function of the Gaaa, with domain Sit(G)×$\mathbb{P}$(N)×Pc(Sit(G)), where $\mathbb{P}$(N) is the power set of N, and range $\mathbb{R}$+ (the set of reals $x \geq 0$) such that:
1) V(S,C,P)≥0; 2) if C= Ø or if C$\notin$Co(Sit(G)) then V(S,C,P)=0.

Obviously V(S,C,P) should be interpreted as the (numerical) value, or 'the amount of utility' that the members of C can obtain from the game (Owen 1995, p.213) in the particular situation S, given the distribution of coalitions in P. By definition, the utility a coalition can obtain depends on the coalition itself, but also on the situation in which the coalition has been formed, that is, the utility of a coalition depends on the behaviour of the coalition itself together with the behaviour of all the other coalitions that are playing in the same situation.

Let us stress that a coalition may obtain some utility even it abstains or is absent: for instance, some side payment. It may obtain some advantages also if it chooses, in a more or less active way, some alternative that is different from the final result. For instance consider a parliament in which a bill has been just approved. A party may have voted against the bill, and so it has not reached its goal and we can say it has lost from the point of view of the ballot; but it may have gained a very great appreciation from the public, so its amount of utility is very high. This party can be considered the true winner from another point of view. This example shows also that a coalition may obtain the result for which it made its effort, so winning the game, but, in the same time earning less than another coalition, that is obtaining an amount of utility smaller than some other coalition.

By these considerations, in this paper we decided to distinguish the two functions FR and V that have been identified up to now in all the papers on power indices, where simply the function FR is implied. For instance in simple games the function FR coincides with the characteristic function that says who wins. In non-simple games it is (implicitly) assumed that a coalition wins if (and only if) it gains more than the other coalitions, and so the function FR again coincides with the characteristic function. In the classical case the function FR is not very important since it is assumed that everyone tries to win and players have only two alternatives: to join one group or to join the complementary group. These two groups/coalitions can win some amount of money (utility) that can be small or big depending on the situation or on the coalition itself. In classical games players have only two alternatives, i.e. win and loss, that are also the only two possible outcomes and they are never expressed explicitly. In practice the same happens in the other papers where several alternatives are considered.

In the definition above we admit games that are partially defined, i.e. when Sit(G) (that is the set of possible situation of the game) is a proper subset of Sit (the set of all the situations) since there may exist situations that are impossible in the game even if they are possible in theory. On the contrary, up to now games have been defined in every situation, i.e. considering Sit(G)=Sit. The fact that Sit(G)$\subseteq$Sit, may be useful to discard from the context improbable situations (see example 7), or study games obtained from some practical case, when, for instance, some coalition has never been observed. It will be necessary also in the following in defining the classical case. By the characteristic function, i.e. the win, another interesting function can be obtained.

9) Definition

a) Let |X| be the cardinality of the set X.
b) s(C)=|{S: C$\in$Co(S) }$\cap$ Sit(G)|.
c) The medium win of a coalition C is the function Mwc with domain Co(Sit(G)) such that:

$$Mwc(C) = \frac{1}{s(C)} \sum_{P\in Pc(Sit(G))} \sum_{C\in Co(S)} V(S,C,P)$$

The value Mwc(C) is the media of all the possible wins a coalition obtains in all the possible situations. In order to study properties of V, we note that:

10) Example and observations

a) If C_k and D_h are coalitions in S but they choose the two different alternatives A_k, and A_h, then $C_k \cup D_h$ is not a coalition in S, and so V(S,C_k,P), V(S, D_h,P) are surely greater than V(S,$C_k \cup D_h$,P), since V(S,$C_k \cup D_h$,P)=0.
b) Consider the 'classical' simple majority wvg with only two alternatives A_1, and A_2, and with voters A, B, C, and D, whose weights are respectively (2,2,3,4). If D is absent, and the situation is S=(A_1, A_1, A_2, Ø), then the set {A,B} is a coalition and is winning in S. But if D is present in the game and abstains, the situation is S'=(A_1, A_1, A_2, A) and the set {A,B,D} is a coalition in S' but is not winning in S'.

This example shows that we cannot have strong forms of monotonicity. Generalizing the classical setting we define:

11) Definition

V(S,C,P) is monotone on coalitions in the situation S if: given two coalitions C and D in S, if C $\cup$D is a coalition in S, then for every P:
V(S,C$\cup$D,P)$\geq$ V(S,C,P), V(S,D,P).
If V is monotone on coalitions in S, we obtain that: if coalitions C and D (in S) are such that C $\subseteq$D, then V(S,C,P)$\leq$V(S,D,P).

3. Classical Case

In this section, we discuss what happens in the classical case in which we have only two possible and so complementary alternatives A_1, and A_2, and voters cannot abstain or be absent. In this section, we denote these two alternatives by 0 and 1: remember we have no abstention and so no possible misunderstanding for the symbol 0. We define:

12) Definition and observations

If we have only two alternatives respectively 0 and 1 and we have no possibility of abstention and absence:

a) if $S=(s_1, \dots , s_n)$, then by $-S= (s'_1, \dots , s'_n)$ we denote the dual of S, that is the situation such that $s'_i=0$ if $s_i=1$ and vice versa.
b) If C is a coalition in S, then by –C=N-C we denote the complement of C, and we note that C and -C are coalitions both in S and –S.

13) Observation

In the situation –S we have that all the players do "the opposite" of what they are doing in the situation S.
If we have only two alternatives, we note also that a partition in coalitions is of the following kind: $P=(C_0,C_1,C_2)$, where C_0 is 0-active in P, C_1 is 1-active in P, and C_2 is the coalition of players that do not want to take a side. Moreover, if the players cannot abstain or be absent, then C_2 is empty, and so a partition in coalitions in this case is of the following kind: $P=(C_0,C_1)$.

By this, suppose again that we have only two alternatives (0 and 1) and no possibility of abstention or absence; we give:

14) Definition

a) If $P=(C_0,C_1)\in Pc(S)$, then –P stands for (C'_0,C'_1), where: $C'_0=C_1$ and $C'_1=C_0$. I.e. C_0 is 0-active in P and 1-active in -P, and analogous for C1. We have that -PÎPc(-S).
b) The Gaaa G is classical if:
   b.1) AL=Out(G) and |AL|=2;
   b.2) Sit(G)=$\{S{:}\,\forall i\in N\; s_i\in\{0,1\}\}$;
   b.3) for every S, C and P: V(S,C,P) = V(-S,C,-P).

15) Observation

Suppose that in the game G we have only two alternatives and no possibility of abstention or absence, then the set of alternatives is simply AL= {0,1}. If the set of voters is N={1, … , n}, by definition it follows that the set Sit(G) is the set of all n-tuples of 0 and 1. Fixed a situation S, let be 1S={X∈N: sx=1 in S}, while 0S= {X∈N: sx=0 in S}. Then 0S =N-1S, and it is easy to prove that Sit(G) is (isomorphic to) the set of all possible pairs (1S, 0S), and also that the two sets {0S: S∈Sit} and {1S: S∈Sit} are isomorphic to P(N) (the power set of N).

16) Example

If N= {A, B} and AL={0,1}, then Sit={(0,0), (1,0), (0,1), (1,1) }. If S=(1,0) then 0S={B}, and 1S={A}, and the set {0S: S∈Sit}= { {A,B}, {B}, {A}, Ø}. Consider C={A}, there are only two situations in which {A} is a coalition in S: indeed {S: {A}∈Co(S)}={(1,0), (0,1)} and we have (1,0)= -(0,1), that is the situation (0,1) is the dual of the situation (1,0), and {A} is 1-active in (1,0) and 0-active in (0,1).

By the assumptions in points 12 and 14, the observation in 13, and generalizing the preceding example, we have:

17) Observations

If the Gaaa is classical (so in particular we have no abstention or absence), then:
a) given a situation S, we have that:
   a.1) |Co(S)|=2; i.e. S determines just one pair of coalitions C_0, C_1, such that: $C_0=\{i\in N: s_i=0\}$, and $C_1=\{i\in N: s_i=1\}$. We have that $C_0=N-C_1$.

a.2) |Pc(S)|=1; i.e. S determines just one partition in coalitions, that is: P= (C_0,C_1).

b) A set $C \subseteq N$ determines just one situation and one partition in which C is 0-active:

b.1) the situation $S^0c=\{s_1, \ldots, s_n\}$, such that: $s_i=0$ « $i \in C$, and 1 otherwise; and

b.2) the partition $P^0c=(C,N-C)$.

c) Analogously C determines just one situation and one partition in which it is 1-active.

d) So s(C)=2.

e) If we have a partition $P=(C_0,C_1)$, then $C_0=N-C_1$. So P determines:

e.1) just one situation $S=\{s_1, \ldots, s_n\}$ such that: $s_i=0$ « $i \in C_0$, and $s_i=1 \leftrightarrow i \in C1$;

e.2) just one pair of coalitions.

f) By the preceding points, and point 14.b.3), we obtain that V=Mwc. Moreover, by points a)--c) here above we can write simply V(C) for V(S,C,P), and V corresponds to the classical characteristic function of a cooperative game. In different words, we can say that the win of C depends only on the coalition C itself and not on the situation or on the behaviour of the other voters.

Note that without condition V(S,C,P) = V(-S,C,-P), it is not possible to completely recover the classical case, since the function V depends not only on coalitions but, essentially, on the situation too as in the general case.

18) Definition

If the Gaaa is classical:

a) if in the situation S it happens that FR(S,P)=0, the set $C=\{X \in N: sx=0\}$ is a coalition in S and is said to be a winning coalition in the goal 0. Analogously if FR(S,P)=1;

b) if $V(S,C,P) \geq V(S,C',P)$ for every $C' \in P$, then we say that C is winning in value in S.

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